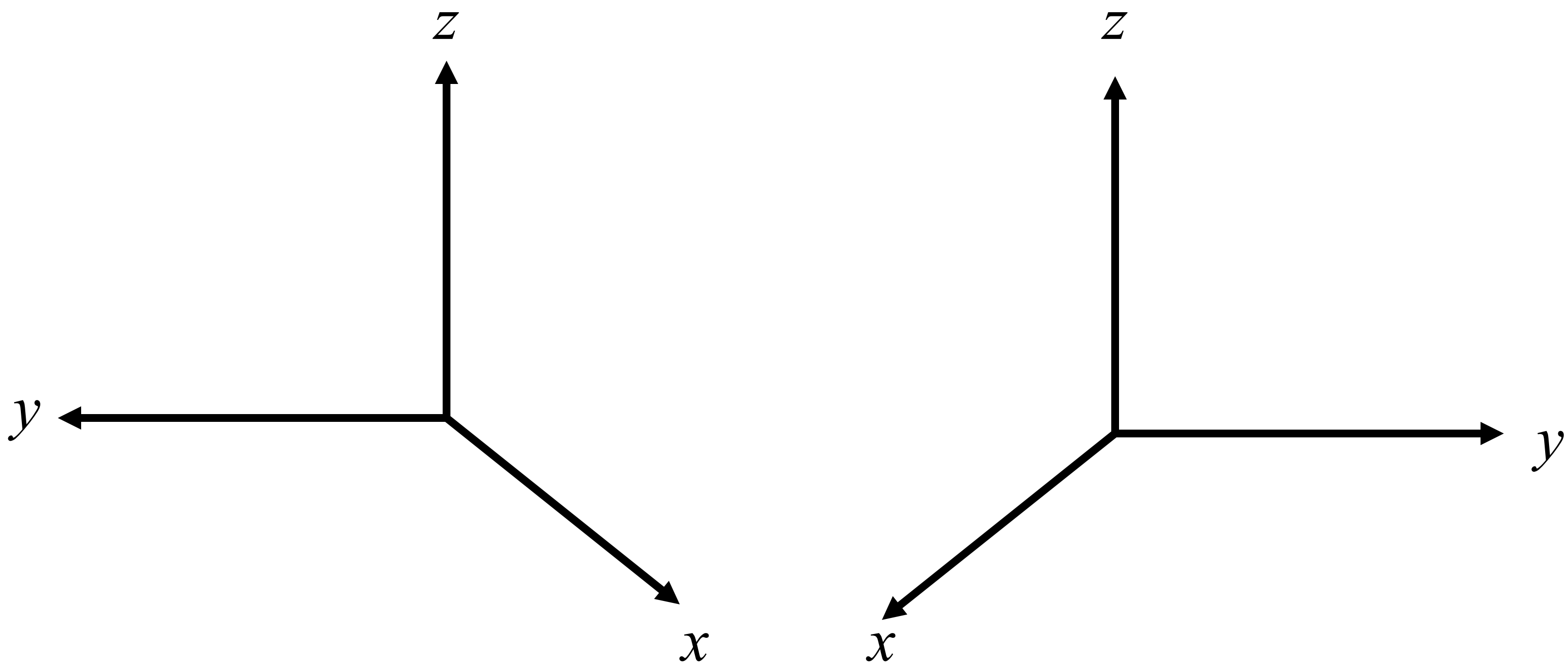
Geometric Transformation

- World, Local, and Viewing Coordinate systems.
- Changing coordinate system transformations
- Affine Transformations (scaling, translation, rotation).
- Structure-deforming transformations (tapering, twisting, bending).

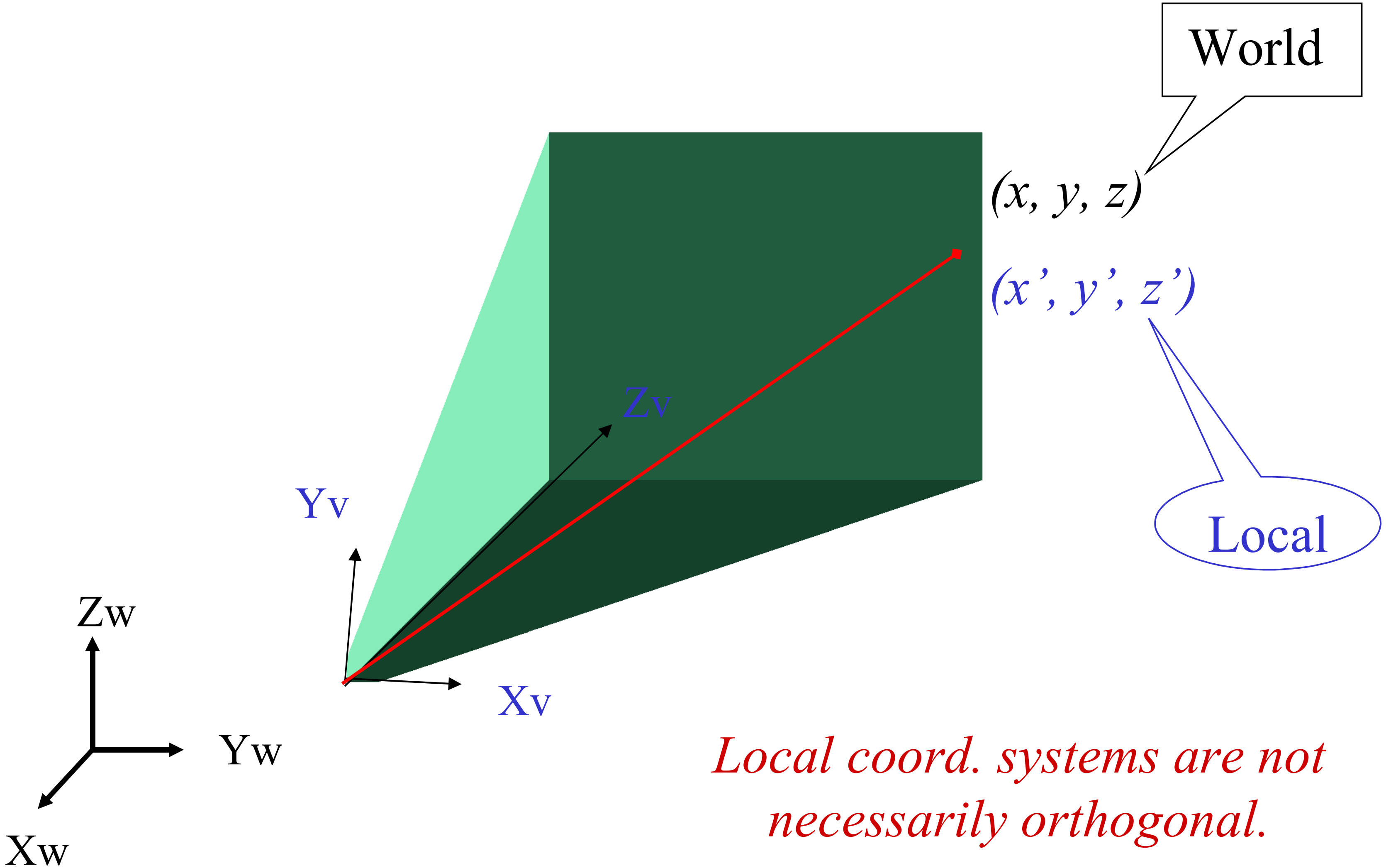
Three-dimensional Coordinate Systems



Left-handed System

Right-handed System

World and Local Coordinate Systems

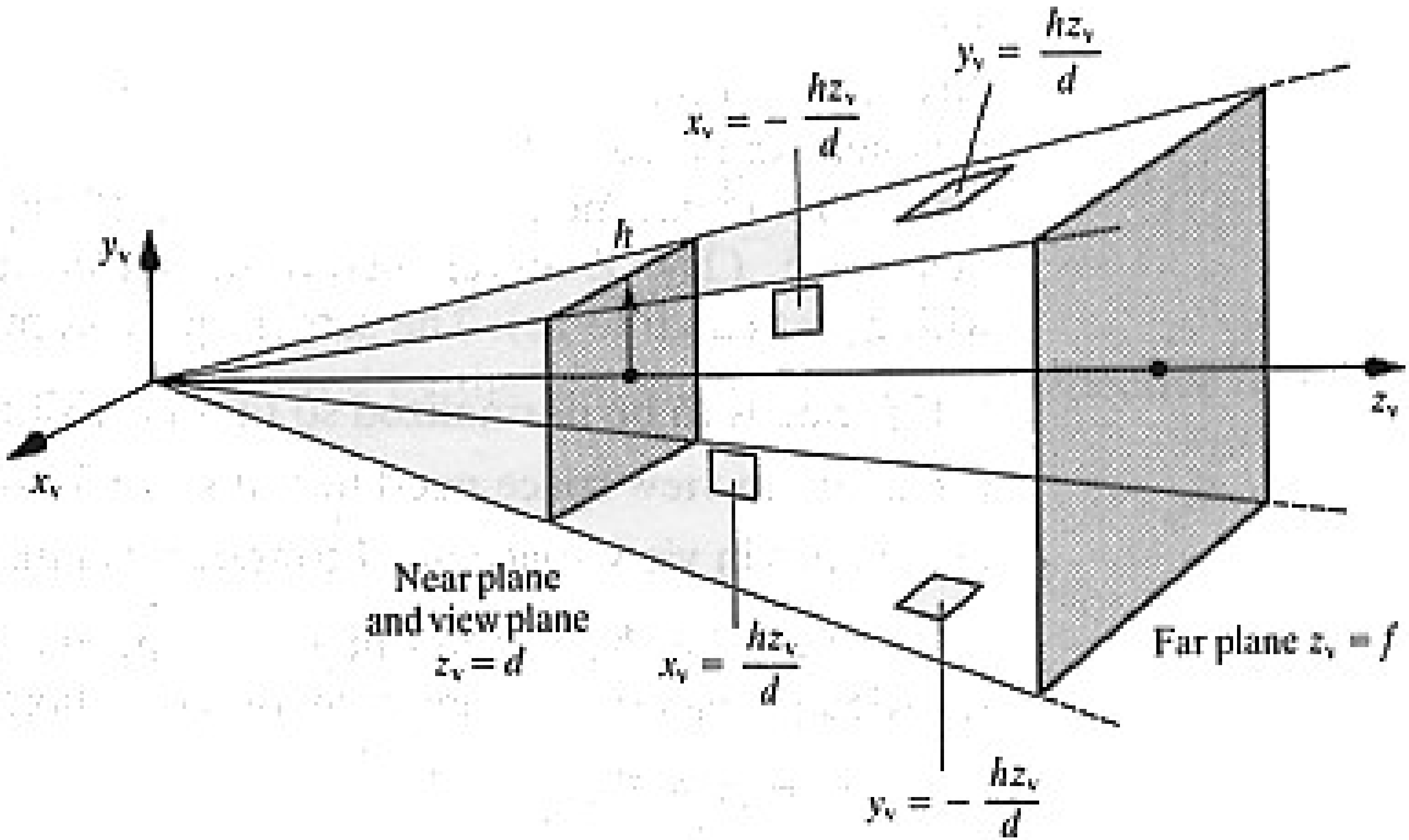


Local coord. systems are not necessarily orthogonal.

Viewing Coordinate Systems

*A viewing coordinate system is a **local** coordinate system defined with reference to the **view point --** a camera, the eyes etc. Viewing coordinate systems usually are "bounded" in a "view volume" (a polyhedron).*

Viewing Coordinate System Example



Affine Transformation

Transform a set of vertices or three-dimensional points belonging to an object into another set of points using a *linear transformation*.

$V' = V + D$ ***D***: displacement

$V' = VS$ ***S***: scaling

$V' = VR$ ***R***: rotation

Homogeneous Coordinates

In a homogeneous coordinate system, a 3D point/vertex $V(x, y, z)$ is represented as:

$$V(X, Y, Z, w)$$

For any scale factor $w \neq 0$

$$x = X/w$$

$$y = Y/w$$

$$z = Z/w$$

Translation

$$[x' \quad y' \quad z' \quad 1] = [x \quad y \quad z \quad 1]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ T_x & T_y & T_z & 1 \end{bmatrix}$$

$$x' = x + T_x$$

$$y' = y + T_y$$

$$z' = z + T_z$$

Scaling

$$V' = VS$$

$$S = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x' = xS_x$$

$$y' = yS_y$$

$$z' = zS_z$$

Uniform scaling: $S_x = S_y = S_z$

Rotation

Need to specify a reference (which axis) to rotate about!

Transformation matrices for *counterclockwise* rotation about X, Y, and Z axes are:

$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y = \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

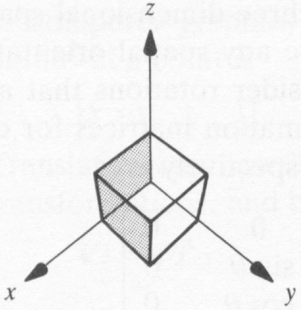
$$R_z = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

General Transformation Matrix

Rotation and Scaling

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ t_x & t_y & t_z & 1 \end{bmatrix}$$

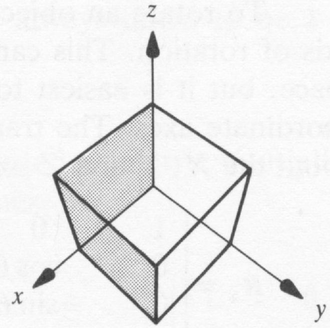
Translation



(a)

Identity

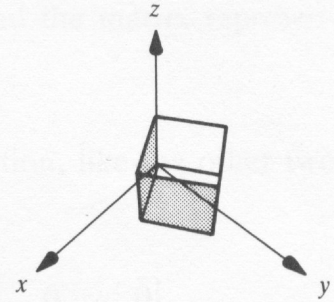
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



(b)

Uniform scaling

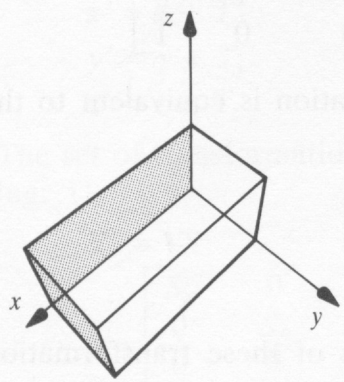
$$\begin{bmatrix} 1.5 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & 0 & 1.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



(c)

Rotation (*z*-axis)

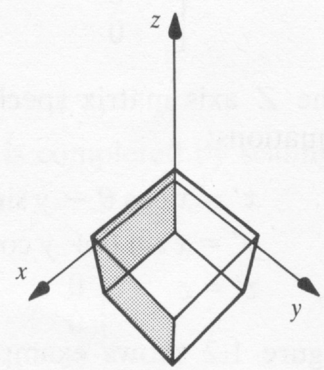
$$\begin{bmatrix} 0.866 & 0.5 & 0 & 0 \\ -0.5 & 0.866 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



(d)

X scaling

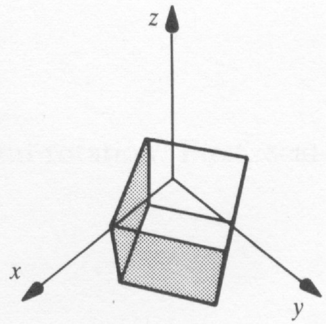
$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



(e)

Translation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 500 & 500 & 0 & 1 \end{bmatrix}$$



(f)

Translation and rotation

$$\begin{bmatrix} 0.866 & 0.5 & 0 & 0 \\ -0.5 & 0.866 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 430.7 & 60.3 & 0 & 1 \end{bmatrix}$$

Concatenation of Transformation Matrices

$$V' = VM_1$$

$$V'' = V'M_2$$

$$V'' = VM_1M_2 = VM$$

$$M = M_1M_2$$

Note: translations are commutative but rotations are not.

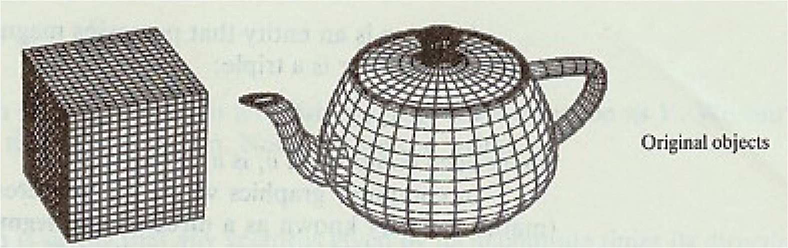
$$T_1T_2 = T_2T_1 \text{ but } R_1R_2 \neq R_2R_1$$

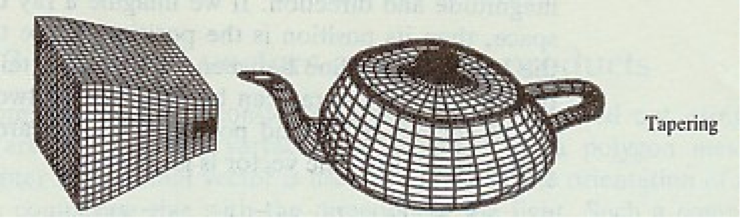
Structure-Deforming Transformation

Tapering

$$X = rx$$
$$Y = ry$$
$$Z = z$$

$$r = f(z)$$



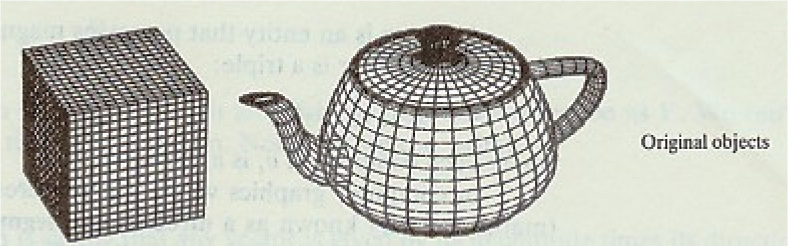


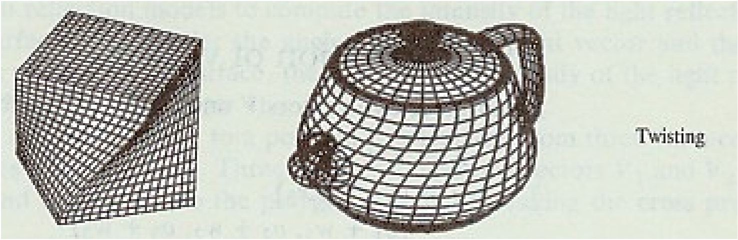
Structure-Deforming Transformation

Twisting

$$X = x\cos\theta - y\sin\theta$$
$$Y = x\sin\theta + y\cos\theta$$
$$Z = z$$
$$\theta = f(z)$$

f'(z) is the rate of twist per unit length along Z.





Structure-Deforming Transformation

Bending

Assume a bending region along Y

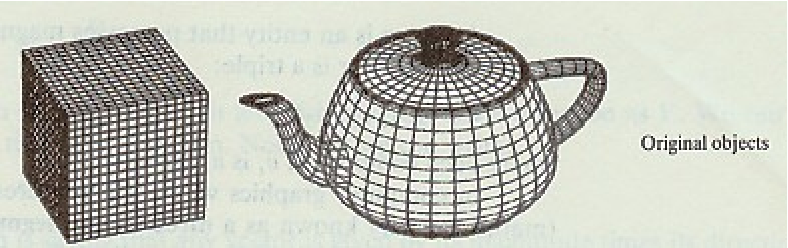
$$y_{min} \le y \le y_{max}$$

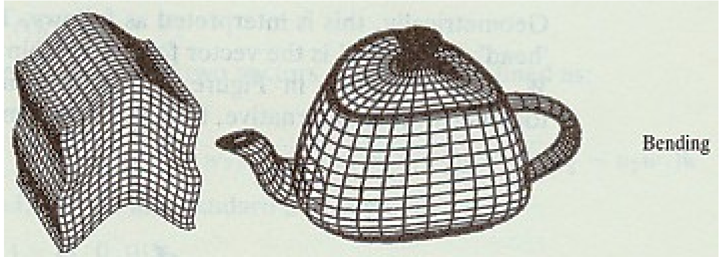
1/k: bending radius of curvature

$y = y_0$: center of bending

$\theta = k(y' - y_0)$: bending angle

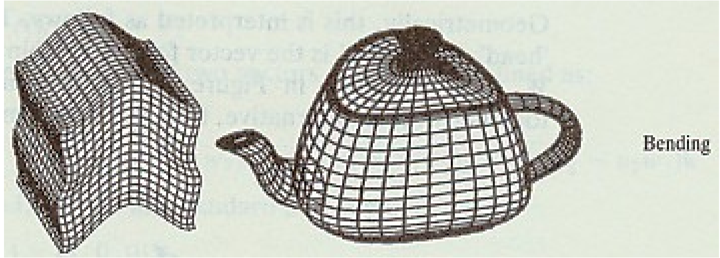
$$y' = \begin{cases} y_{min} & y \leqslant y_{min} \\ y & y_{min} \leqslant y_{max} \\ y_{max} & y > y_{max} \end{cases}$$





Structure-Deforming Transformation

Bending (cont.)



$$X = x$$

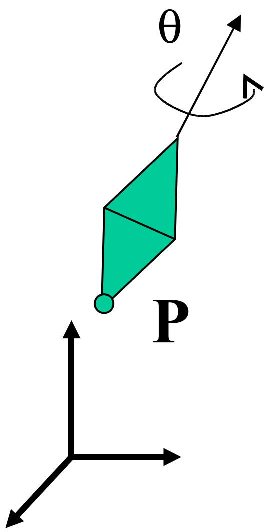
$$Y = \begin{cases} -\sin\theta(z - 1/k) + y_0 & y_{min} \leqslant y \leqslant y_{max} \\ -\sin\theta(z - 1/k) + y_0 + \cos thata(y - y_{min}) & y < y_{min} \\ -\sin\theta(z - 1/k) + y_0 + \cos\theta(y - y_{max}) & y > y_{max} \end{cases}$$

$$Z = \begin{cases} -\sin\theta(z - 1/k) + y_0 & y_{min} \leqslant y \leqslant y_{max} \\ -\sin\theta(z - 1/k) + y_0 + \cos thata(y - y_{min}) & y < y_{min} \\ -\sin\theta(z - 1/k) + y_0 + \cos\theta(y - y_{max}) & y > y_{max} \end{cases}$$

Changing Coordinate Systems Transformations

Transformation between coordinate systems is the inverse of the transformation that takes the old axes to the new axes within the current coordinate system.

Inter-Coord. Transformation

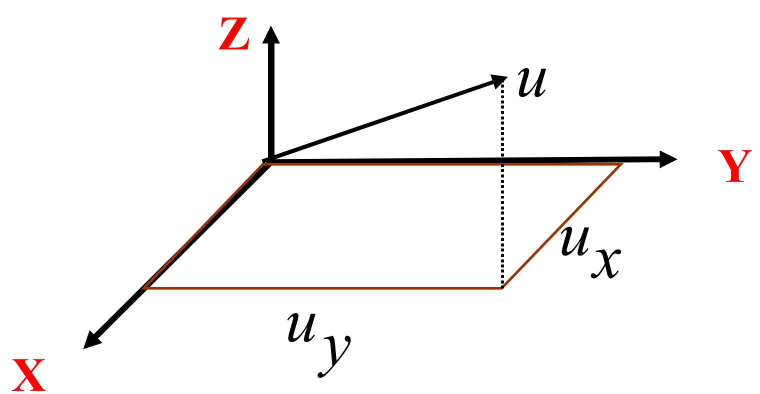


$$\vec{u} = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix} \quad |\vec{u}| = 1$$

Rotate the object about u

1. Translate ***P*** to origin

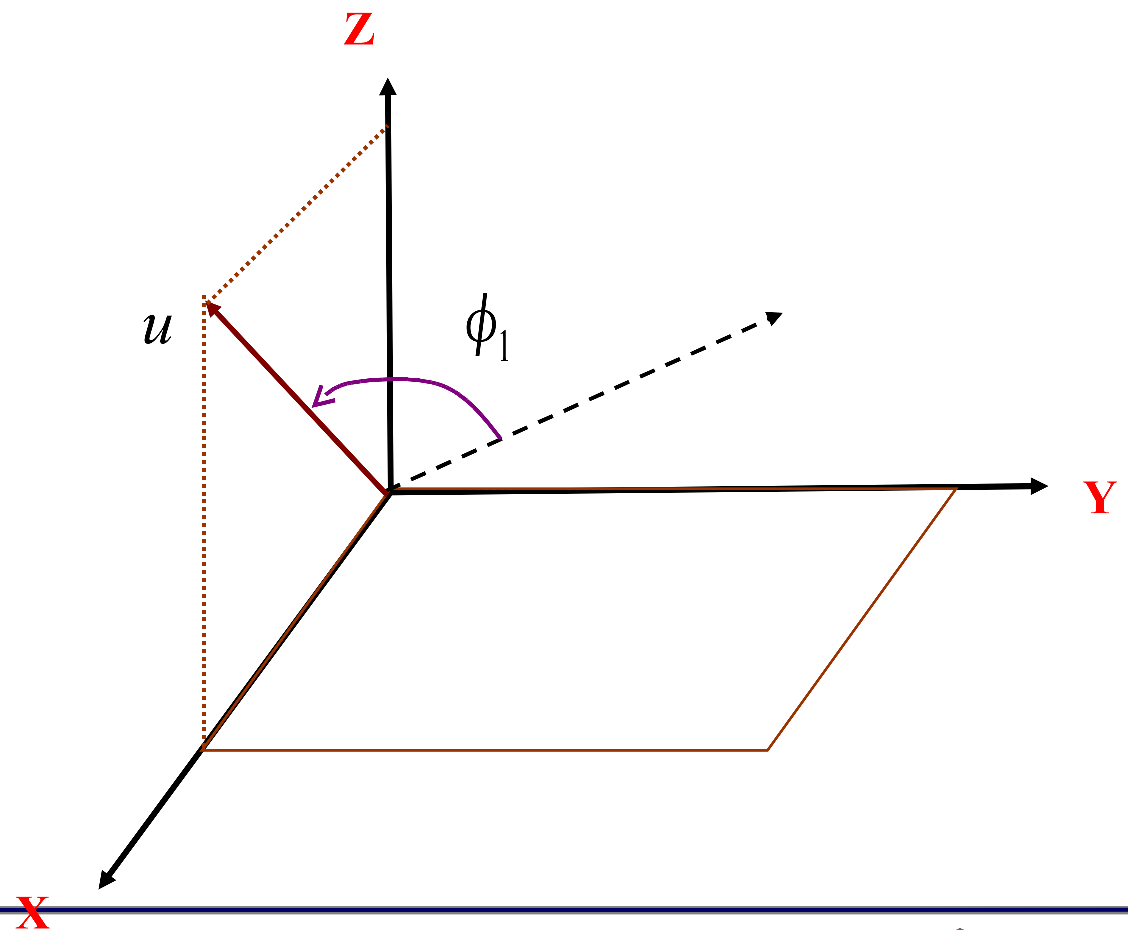
$$\boldsymbol{T}(-P_x, -P_y, -P_z)$$



Inter-Coord. Transformation (cont.)

2. Rotate about *X* until u lies in *X-Z* plan

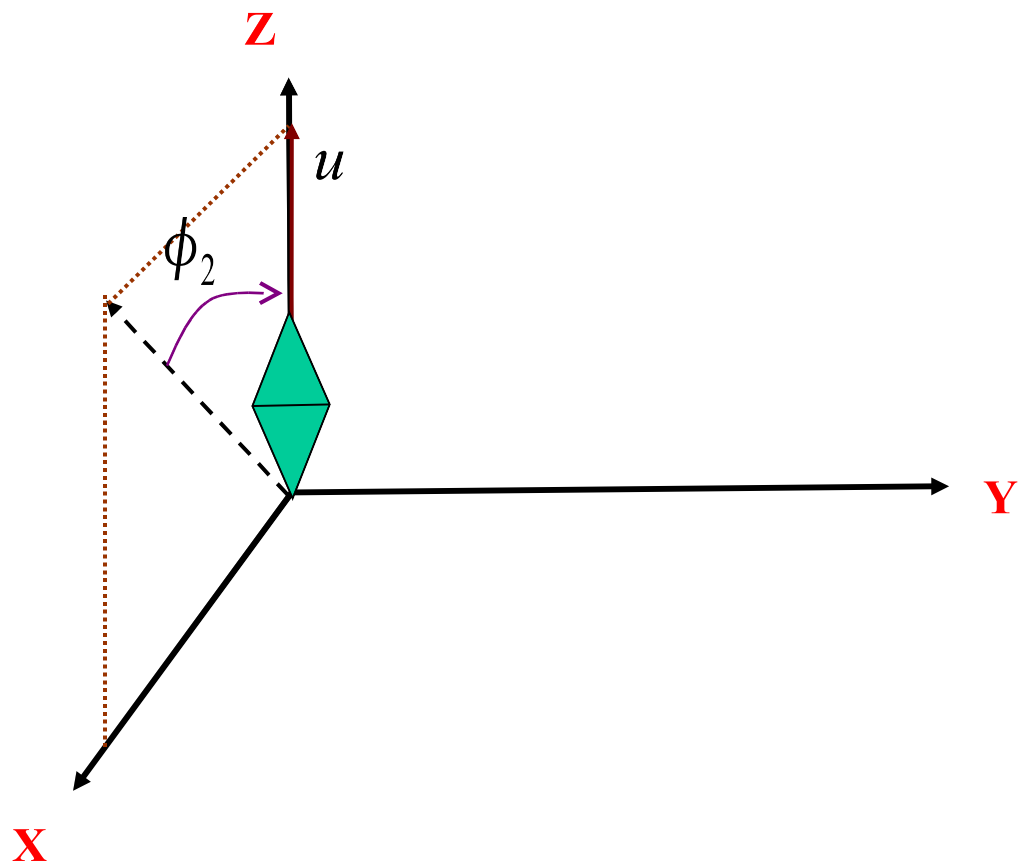
$$R_x(\phi_1)$$



Inter-Coord. Transformation (cont.)

3. Rotate about Y until u is identical to Z.

$R_y(\phi_2)$



$R_z(\theta)$ 4. Rotate about Z axis for θ .

Inter-Coord. Transformation (cont.)

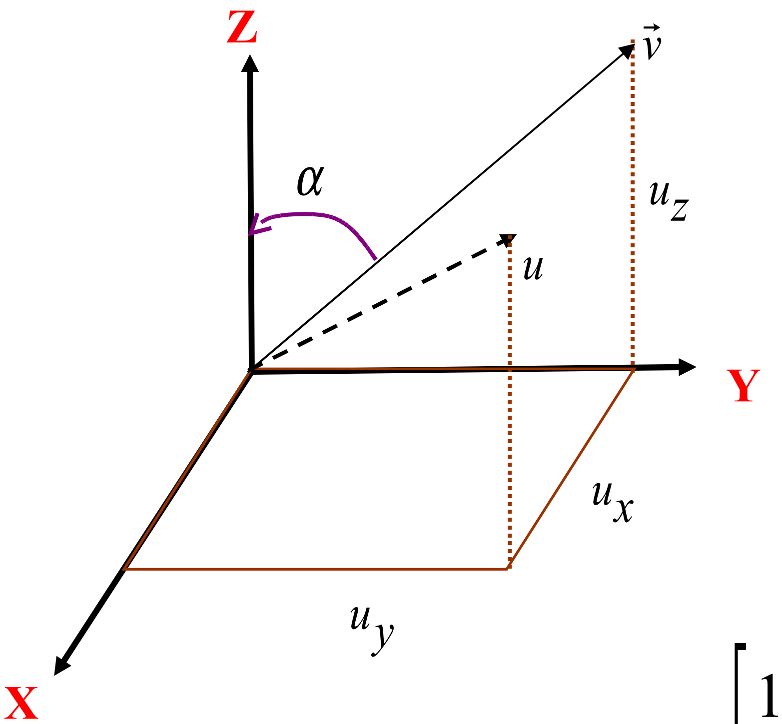
5. “Unrotate” for step 3 and 2 and “untranslate” for step 1.

$$R_y^{-1} R_x^{-1} T^{-1}$$

The final combined transformation matrix will be:

$$M = TR_x(\phi_1) R_y(\phi_2) R_z(\theta) R_y^{-1}(\phi_2) R_x^{-1}(\phi_1) T^{-1}$$

Inter-Coord. Transformation (cont.)



$$v = \sqrt{u_z^2 + u_y^2}$$

is the projection of $\vec{u}$ in *Y-Z*

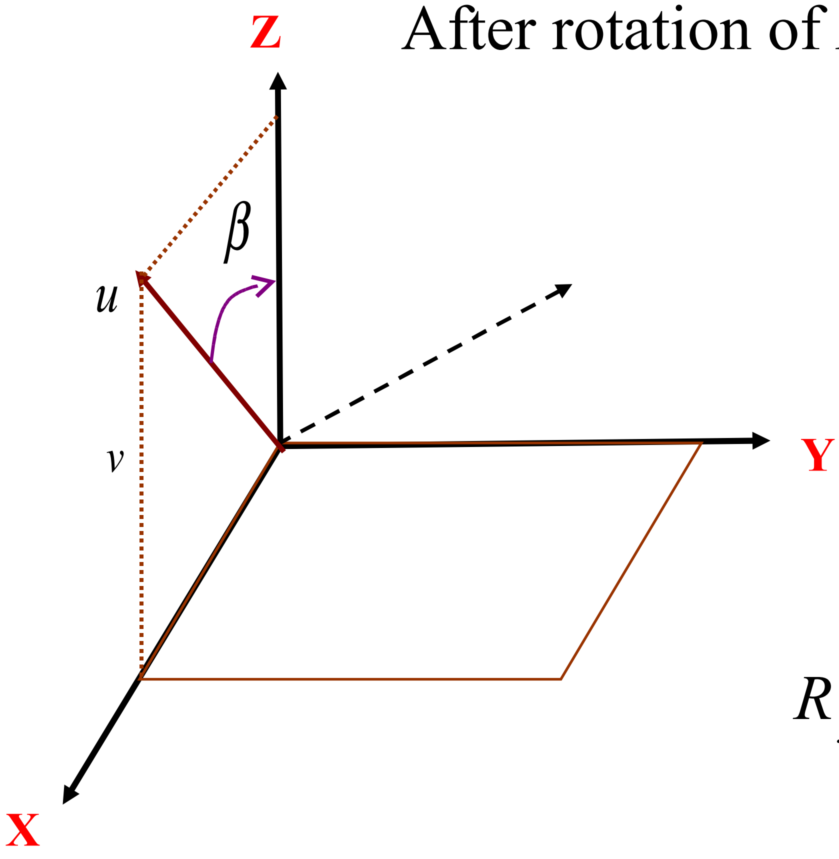
$$\alpha = \phi_1$$

$$\cos\alpha = u_z/v;\ \sin\alpha = u_y/v$$

$$R_x(\phi_1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & u_z/v & u_y/v & 0 \\ 0 & -u_y/v & u_z/v & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inter-Coord. Transformation (cont.)

After rotation of R_x



$$\beta = \phi_2$$

$$\cos\beta = v\,;\quad \sin\beta = u_x$$

$$R_y(\phi_2) = \begin{bmatrix} v & 0 & u_x & 0 \\ 0 & 1 & 0 & 0 \\ -u_x & 0 & v & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inter-Coord. Transformation Example